

SOME CONJECTURES IN ELEMENTARY NUMBER THEORY

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ABSTRACT. We announce a number of conjectures associated with and arising from a study of primes and irrationals in \mathbb{R} . All are supported by numerical verification to the extent possible.

THE CONJECTURES

Bhargava factorials. For definitions and basic results dealing with Bhargava's factorial functions we refer to [3], [4], [5] and [8]. Briefly, let $X \subseteq \mathbf{Z}$ be a finite or infinite set of integers. Following [5], one can define the notion of a p -ordering on X and use it to define a set of generalized factorials of the set X inductively. By definition $0!_X = 1$. Whenever p a prime, we fix an element $a_0 \in X$ and, for $k \geq 1$, we select a_k such that the highest power of p dividing $\prod_{i=0}^{k-1} (a_k - a_i)$ is minimized. The resulting sequence of a_i is then called a p -ordering of X . As one can gather from the definition, p -orderings are not unique, as one can vary a_0 . On the other hand, associated with such a p -ordering of X we define an associated p -sequence $\{\nu_k(X, p)\}_{k=1}^{\infty}$ by

$$\nu_k(X, p) = w_p\left(\prod_{i=0}^{k-1} (a_k - a_i)\right),$$

where $w_p(a)$ is, by definition, the highest power of p dividing a (e.g., $w_2(80) = 16$). One can show that although the p -ordering is not unique the associated p -sequence is independent of the p -ordering used. Since this quantity is an invariant it can be used to define generalized factorials of X by setting

$$k!_X = \prod_p \nu_k(X, p), \quad (1)$$

where the (necessarily finite) product extends over all primes p .

Definition 1. [12]. *An abstract (or generalized) factorial is a function $!_a : \mathbb{N} \rightarrow \mathbb{Z}^+$ that satisfies the following conditions:*

$$(1) \quad 0!_a = 1,$$

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- (2) For every non-negative integers n, k , $0 \leq k \leq n$ the generalized binomial coefficients

$$\binom{n}{k}_a := \frac{n!_a}{k!_a (n-k)!_a} \in \mathbb{Z}^+,$$

- (3) For every positive integer n , $n!$ divides $n!_a$.

It is easy to see that the collection of all abstract factorials forms a commutative semigroup under ordinary pointwise multiplication. In fact, it is easy to see that Bhargava's factorial function is an abstract factorial. (Indeed, Hypothesis 1 of Definition 1 is clear by definition of the factorial in question. Hypothesis 2 of Definition 1 follows by the results in [5].)

The context of these first three conjectures is the construction in [5] as applied to the ring of integers. In this case, the factorial function for the set of rational primes

$$\mathbb{P} = \{2, 3, 5, 7, 11, \dots\}$$

is given by [5]

$$n!_{\mathbb{P}} = \prod_p p^{\sum_{m=0}^{\infty} \lfloor \frac{n-1}{p^m(p-1)} \rfloor}. \quad (2)$$

We call this simply the B-factorial for the set under consideration. In the sequel, the statement “For every $n \geq 1$ ” means “for every integer $n \geq 1$ for which the factorials are defined”.

Let $\mathbb{P}_2 \subset \mathbb{P}$ denote the subset of all twin primes, i.e., those primes of the form $p, p+2$ as usual. Let $n!_{\mathbb{P}_2}$ denote the B-factorial of the set \mathbb{P}_2 . In the following conjectures the notation $w_p(n)$ is used to identify the highest power of p that divides n . So, for example, if n has the representation $n = 2^{a_1} \alpha$ and $(\alpha, 2) = 1$, then $w_2(n) = 2^{a_1}$.

Conjecture 1. For every $n \geq 1$,

$$\frac{n!_{\mathbb{P}_2}}{n!_{\mathbb{P}}} = 2 w_2(n).$$

In analogy with the preceding we let $\mathbb{P}_3 \subset \mathbb{P}$ denote that subset of all prime triplets of the form $p, p+2, p+6$. Let $n!_{\mathbb{P}_3}$ denote the B-factorial of the set \mathbb{P}_3 .

Conjecture 2. For every $n \geq 1$,

$$\frac{n!_{\mathbb{P}_3}}{n!_{\mathbb{P}}} = \begin{cases} 3! w_2(n) w_3(n), & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Next, let $\mathbb{P}_4 \subset \mathbb{P}$ denote that subset of all prime quadruplets written in the form $p, p+2, p+6, p+8$. Since $p, p+2$ and $p+6, p+8$ are both twin primes we can view $\mathbb{P}_4 \subset \mathbb{P}_2$, and so we must have $n!_{\mathbb{P}_2} | n!_{\mathbb{P}_4}$, by [5], Lemma 13]. In fact, we claim that,

Conjecture 3. For every $n \geq 1$,

$$\frac{n!_{\mathbb{P}_4}}{n!_{\mathbb{P}_2}} = \begin{cases} 3 w_3(n), & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

These three conjectures have been verified using Crabbe's algorithm [10] to the limits available by the hardware. For motivation see [12].

Prime number inequalities. Now let p_n denote the n -th prime. Then, see [12],

Conjecture 4.

$$p_n \geq p_k + p_{n-k-1}, \quad 1 \leq k \leq n-1, \quad (3)$$

and all $n \geq 2$.

The validity of this conjecture implies that the function $f : \mathbb{N} \rightarrow \mathbb{Z}^+$,

$$f(n) = \begin{cases} 1, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ p_{n-1}!, & \text{if } n \geq 2. \end{cases}$$

is an abstract factorial. Thus, if true, it would follow from the results in [12] that for any abstract factorial $n!_a$, the quantity $\sum_{n \geq 1} 1/n!_a f(n) \notin \mathbb{Q}$.

Apéry numbers. We define the Apéry numbers A_n, B_n recursively, as usual, by setting $A_0 = 1, A_1 = 5$; $B_0 = 0, B_1 = 6$ whose general terms are given by the recurrence relations

$$A_{n+1} = (P(n)A_n - n^3 A_{n-1})/(n+1)^3,$$

and

$$B_{n+1} = (P(n)B_n - n^3 B_{n-1})/(n+1)^3,$$

where $P(n)$ is the polynomial

$$P(n) = 34n^3 + 51n^2 + 27n + 5.$$

In a singular argument Apéry [1] showed that $B_n/A_n \rightarrow \zeta(3)$ as $n \rightarrow \infty$ where ζ is the usual Riemann zeta function. In addition, he proved that $\zeta(3)$ is irrational (though no explicit formula akin to the one known for the values of ζ at positive even integers was given). More explicit proofs appeared since, e.g., [2], [14], [9] among others. (See [13] for extensions of the series acceleration method found in [Fischler [11], Remarque 1.3] to integer powers of $\zeta(3)$.)

Here we propose using an old irrationality criterion due to Brun [6] (see also [7]) in order to formulate a conjecture that, if true, would give another proof of the irrationality of $\zeta(3)$. Let x_n be a sequence of real numbers and Δ the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$.

Theorem 2. (Brun, [6]) *Let $x_n \in \mathbb{Z}^+$ be an increasing sequence and $y_n \in \mathbb{Z}^+$ be such that $\Delta(y_n/x_n) > 0$. If*

$$\delta_n \equiv \Delta(\Delta y_n / \Delta x_n) < 0, \quad (4)$$

then y_n/x_n converges to an irrational number.

Although Brun claimed later [7] that “... this theorem is simple but unfortunately not very useful” we show that perhaps it may be used to prove the irrationality of $\zeta(3)$.

The idea is as follows: It is known that the sequence A_n of Apéry numbers is an increasing sequence of positive integers [9] and although the B_n is not necessarily a sequence of integers, the weighted sequence $e_n B_n$ is such a sequence where

$e_n = 2 \cdot (\text{lcm}\{1, 2, \dots, n\})^3$, [9]. In addition, the sequence $B_n/A_n = e_n B_n / e_n A_n$ is increasing, [9] and it is easily proved that the sequence $e_n A_n$ is increasing as well.

Thus, setting $x_n = e_n A_n$ and $y_n = e_n B_n$ we see that the requirements x_n is increasing and y_n/x_n increasing are met in Theorem 2 (all sequences being positive and all integers). We anticipate the following

Conjecture 5. *There is an unbounded subsequence of positive integers $n_k \rightarrow \infty$ such that $\delta_{n_k} < 0$.*

Since it is known that y_n/x_n increases to $\zeta(3)$, clearly y_{n_k}/x_{n_k} does the same for any subsequence. Hence, an affirmative answer to the previous conjecture implies the irrationality of $\zeta(3)$ by Brun's irrationality theorem, Theorem 2. The numerical evidence seems to point to a stronger conjecture however. Indeed, it appears as if

Conjecture 6. *For every integer $N \geq 2$, there is an $n \in \mathbb{Z}^+$ such that all*

$$\delta_n, \delta_{n+1}, \delta_{n+2}, \dots, \delta_{n+N} < 0.$$

Of course this result, if true, implies the previous conjecture.

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